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Optical amplifier performance in digital optical communication systems

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Contents Optical amplifiers are important building blocks in nowaday's optical communication system. Their noise behavior differs considerably from that of electrical amplifiers. Theory is reviewed and supplemented by examples that allow to estimate the performance of optical transmission systems. In particular, a correct optical noise figure definition is explained.

1 Introduction

Optical communication via glass fibers is growing at extraordinary rates. Traditional electrical data regenerators are replaced, where possible, by optical amplifiers. At some place, however, the optical signal has to be detected and regenerated electrically. Optical amplification has been originally described more than 40 years ago [1]. Photoelectron statistics are also described in [2] but these excellent treatments seem to be not widely known today. In particular, a noise figure definition exists [3] which does not permit exact calculation of the resulting receiver sensitivity. In a recent publication [4] this was corrected. This has prompted the author, who has used part of this manuscript since 1996 in lectures at the Univ. Paderborn, Germany, to review this subject. Practical examples are also given.

2 Photon number distribution

All random variables (RVs) in this paper are nonnegative, except for Gaussian RVs. The interaction of photons with matter is governed by stimulated emission, absorption, and spontaneous emission. For infinitesimal time increments $\mathrm{d}t$ the probability $P(n, t+\mathrm{d}t)$ to find n photons in a medium at time $t+\mathrm{d}t$ depends on the probabilities to find $n-1$, n, or $n+1$ photons at time t, and the conditional transition probabilities from one of these numbers to n:

$$P(n, t+\mathrm{d}t) = P(n|n)P(n,t) + P(n|n-1)P(n-1,t) + P(n|n+1)P(n+1,t) \quad . \tag{1}$$

Other photon numbers need not be considered because for $\mathrm{d}t \to 0$ the probability of multiple emission/absorption events tends towards zero faster than that of single events. This master equation of photon statistics can be written in differential form as

$$\frac{\mathrm{d}P(n,t)}{\mathrm{d}t} = -(n(a+b)+c)P(n,t) + ((n-1)a+c) \times P(n-1,t) + (n+1)bP(n+1,t) \quad , \tag{2}$$

where a is the stimulated emission rate, b the absorption rate and c the spontaneous emission rate. If a particular photon statistic is to persist it must fulfill (2) but statistical parameters such as its $\langle n \rangle$ may be time-variable. For example, if there is only attenuation ($b > 0$, $a = c = 0$), a Poisson distribution

$$P(n) = \mathrm{e}^{-\mu_0} \frac{\mu_0^n}{n!} \tag{3}$$

is a solution of (2) if its expectation value $\langle n \rangle = \mu_0(t)$ decays exponentially with time, $\mu_0(t) = \mu_0(0)\mathrm{e}^{-bt}$. This means a Poisson distribution is conserved under pure attenuation.

The moment generating function (MGF) of a discrete RV n is

$$M_n(\mathrm{e}^{-s}) = \langle \mathrm{e}^{-sn} \rangle = \sum_{n=-\infty}^{\infty} P(n)\mathrm{e}^{-sn} \quad , \tag{4}$$

for a continuous random variable x it is

$$M_x(\mathrm{e}^{-s}) = \langle \mathrm{e}^{-sx} \rangle = \int_{-\infty}^{\infty} p_x(x)\mathrm{e}^{-sx}\,\mathrm{d}x \quad . \tag{5}$$

The lower summation index/integration boundary is 0 for nonnegative RVs. With $\mathrm{e}^{-s} = z^{-1}$, Eq. (4) can be inverted by inverse z, while (5) is inverted by inverse Laplace transformation.

Adding statistically independent RVs requires convolution of the corresponding PDFs, or multiplication of the corresponding MGFs.

As suggested by the name moment generating function, the MGF allows to obtain all moments via

$$\left\langle x^k \right\rangle = (-1)^k \left. \frac{\mathrm{d}^k M(\mathrm{e}^{-s})}{(\mathrm{d}s)^k} \right|_{s=0} \quad . \tag{6}$$

Distributions, MGFs, mean values and variances of some discrete and continuous RVs are given in Table 1.

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When a signal passes an optical amplifier the probabilities $P(n,t)$ of the discrete number of photons n is time-variable. We determine the temporal derivative of the corresponding time-variable MGF $M_n(\mathrm{e}^{-s},t)$ [6].

$$\frac{\partial}{\partial t}M_n(\mathrm{e}^{-s},t)=\sum_{n=0}^{\infty}\mathrm{e}^{-sn}\frac{\mathrm{d}P(n,t)}{\mathrm{d}t} \quad . \tag{7}$$

Using (2) we obtain

$$\frac{\partial}{\partial t}M_n(\mathrm{e}^{-s},t)=c(\mathrm{e}^{-s}-1)M_n(\mathrm{e}^{-s},t)-(a-b\,\mathrm{e}^{s}) \times(\mathrm{e}^{-s}-1)\frac{\partial}{\partial s}M_n(\mathrm{e}^{-s},t) \quad . \tag{8}$$

The signal reaches the amplifier input at $t=0$, with a photon distribution $P(n,0)$ and an MGF $M_n(\mathrm{e}^{-s},0)$. The solution of (8) is

$$M_n(\mathrm{e}^{-s},t)=\left(\frac{a-b}{a-b+a(G-1)(1-\mathrm{e}^{-s})}\right)^{c/a} \times M_n\left(\frac{a-b+(bG-a)(1-\mathrm{e}^{-s})}{a-b+a(G-1)(1-\mathrm{e}^{-s})},0\right) \tag{9}$$

with a time-variable (power) gain

$$G=G(t)=\mathrm{e}^{(a-b)t} \quad . \tag{10}$$

This means we know the MGF inside or at the output of an optical amplifier as a function of the MGF at its input, if the term e^{-s} is replaced by a more complicated one.

For a transmitted zero there are no photons at the input,

$$P(n,0)=\begin{cases}1 & n=0\\ 0 & n\neq 0\end{cases}, \quad M_n(\mathrm{e}^{-s},0)=1 \quad . \tag{11}$$

At time t the signal has passed the amplifier and $G=\mathrm{e}^{(a-b)t}$ holds. According to (9) the output photon MGF is

$$M_n(\mathrm{e}^{-s},t)=\left(1+\frac{a}{a-b}(G-1)(1-\mathrm{e}^{-s})\right)^{-c/a} \quad . \tag{12}$$

The dependence on t can now be dropped. If there is just one optical mode stimulated emission is just n times larger than spontaneous emission, hence $c=a$. For N noisy modes

$$N=\frac{c}{a} \tag{13}$$

holds. The spontaneous emission factor is

$$n_{\mathrm{sp}}=\frac{a}{a-b} \quad . \tag{14}$$

The expectation value of the photons in one mode is

$$\mu=n_{\mathrm{sp}}(G-1) \quad . \tag{15}$$

With above substitutions the MGF is

$$M_n(\mathrm{e}^{-s})=(1+\mu(1-\mathrm{e}^{-s}))^{-N} \quad . \tag{16}$$

According to Table 1, the quotient between variance and mean value equals $\mu+1$ and is larger than for the Poisson distribution where it is 1. This is due to the quantum noise of the optical amplifier. It surpasses the quantum noise of a Poisson distribution which is just shot noise. Inverse z transformation of $M_n(z^{-1})$ results in a (central) negative Binomial distribution

$$P(n)=\binom{n+N-1}{n}\frac{\mu^n}{(1+\mu)^{n+N}} \quad . \tag{17}$$

A sufficiently attenuated signal is always Poisson distributed, as will be seen later. For a transmitted one we therefore assume a Poisson distribution at the amplifier input. With $M_n(\mathrm{e}^{-s},0)=\mathrm{e}^{\tilde{\mu}_0(\mathrm{e}^{-s}-1)}$ and (9) we find

$$M_n(\mathrm{e}^{-s},t)=(1+\mu(1-\mathrm{e}^{-s}))^{-N}\mathrm{e}^{\frac{-\mu_0(1-\mathrm{e}^{-s})}{1+\mu(1-\mathrm{e}^{-s})}} \quad , \tag{18}$$

at the amplifier output with $\mu_0=\tilde{\mu}_0 G$. The corresponding photon distribution is a noncentral negative binomial or Laguerre distribution

$$P(n)=\frac{\mu^n}{(1+\mu)^{n+N}}\,\mathrm{e}-\frac{\mu_0}{1+\mu}L_n^{N-1}\left(\frac{-\mu_0}{\mu(1+\mu)}\right) \quad , \tag{19}$$

Table 1. Some discrete and continuous distributions

Discrete distributions	$P(n)(n\geq 0)$	$M_n(\mathrm{e}^{-s})$	$\langle n\rangle$	σ_n^2
Poisson	$\mathrm{e}^{-\mu_0}\frac{\mu_0^n}{n!}$	$\mathrm{e}^{-\mu_0(1-\mathrm{e}^{-s})}$	μ_0	μ_0
Central negative binomial	$\binom{n+N-1}{n}\frac{\mu^n}{(1+\mu)^{n+N}}$	$\frac{1}{(1+\mu(1-\mathrm{e}^{-s}))^N}$	$N\mu$	$N\mu(\mu+1)$
Noncentral negative binomial, Laguerre	$\frac{\mu^n\,\mathrm{e}^{-\frac{\mu_0}{1+\mu}}}{(1+\mu)^{n+N}}L_n^{N-1}\left(\frac{-\mu_0}{\mu(1+\mu)}\right)$	$\frac{\mathrm{e}^{\frac{-\mu_0(1-\mathrm{e}^{-s})}{1+\mu(1-\mathrm{e}^{-s})}}}{(1+\mu(1-\mathrm{e}^{-s}))^N}$	$\mu_0+N\mu$	$N\mu(\mu+1)+(2\mu+1)\mu_0$
Continuous distributions	$p_{\tilde{x}}(\tilde{x})(\tilde{x}\geq 0)$	$M_{\tilde{x}}(\mathrm{e}^{-s})$	$\langle\tilde{x}\rangle$	$\sigma_{\tilde{x}}^2$
Constant	$\delta(\tilde{x}-\tilde{\mu}_0)$	$\mathrm{e}^{-\tilde{\mu}_0 s}$	$\tilde{\mu}_0$	0
Central χ_{2N}^2, Gamma	$\frac{1}{\Gamma(N)}\tilde{\mu}^N\tilde{x}^{N-1}\,\mathrm{e}^{-\tilde{x}/\tilde{\mu}}$	$\frac{1}{(1+\tilde{\mu}s)^N}$	$N\tilde{\mu}$	$N\tilde{\mu}^2$
Noncentral χ_{2N}^2	$\frac{\tilde{x}^{\frac{N-1}{2}}\,\mathrm{e}^{-\frac{\tilde{\mu}_0+\tilde{x}}{\tilde{\mu}}}}{\tilde{\mu}_0^{\frac{N-1}{2}}\tilde{\mu}}I_{N-1}\left(\frac{2\sqrt{\tilde{x}\tilde{\mu}_0}}{\tilde{\mu}}\right)$	$\frac{\mathrm{e}^{\frac{-\tilde{\mu}_0 s}{1+\tilde{\mu}s}}}{(1+\tilde{\mu}s)^N}$	$\tilde{\mu}_0+N\tilde{\mu}$	$N\tilde{\mu}^2+2\tilde{\mu}\tilde{\mu}_0$

with Laguerre polynomials defined as

$$L_n^\alpha(x) = \frac{1}{n!}\ e^x x^{-\alpha}\ \frac{d^n}{dx^n}\left(e^{-x}x^{n+\alpha}\right) = \sum_{m=0}^{n}(-1)^m \binom{n+\alpha}{n-m}\frac{x^m}{m!}\ . \tag{20}$$

There is an important adding property: two statistically independent RVs n_1, n_2 with the same noise μ and (central or noncentral) negative binomial distributions are added to form a new RV n. It obeys a new negative binomial distribution with $\mu_0 = \mu_{0,1} + \mu_{0,2}$, $N = N_1 + N_2$. This can be verified by convolving the PDFs or multiplying the MGFs.

3 Noise figure

It is useful to divide the output noise photon number μ per mode by the amplification G. Thereby one obtains ficticious input-referred noise which would be amplified by a noise-free amplifier to become the observed output noise. This is the additional noise figure

$$F_z = \frac{\mu}{G} = \left(1 - G^{-1}\right)n_{sp}\ . \tag{21}$$

The noise figure itself is

$$F = 1 + F_z = 1 + \left(1 - G^{-1}\right)n_{sp}\ . \tag{22}$$

This noise figure can be called F_{ASE} [4, 5], where ASE stands for amplified spontaneous emission. Due to $n_{sp} \geq 1$ one finds $F \geq 1$ for amplifiers. An ideal amplifier with $n_{sp} = 1$ and amplification $G \to \infty$ possesses $F = 2$, which can be understood from the following:

- Without amplifier just the amplitude but not the phase of an optical signal can be measured.
- With ideal amplifier and a subsequent power splitter both these quantities can be measured simultaneously.

Let us cascade two optical amplifiers. For a Poisson distribution with mean $\tilde{\mu}_0$ at its input the MGF after the first amplifier (index 1) is

$$M_{n,1}(e^{-s}, t_1) = \left(1 + \mu_1(1 - e^{-s})\right)^{-N} e^{\frac{-\mu_{0,1}(1-e^{-s})}{1+\mu_1(1-e^{-s})}} \tag{23}$$

with $\mu_{0,1} = \tilde{\mu}_0 G_1$. Under the assumption that there is only one optical filter, behind all amplifiers and directly in front of the receiver, the mode number N is identical for both amplifiers. The second amplifier carries the index 2. The MGF at its output is

$$M_{n,2}(e^{-s}, t_2 + t_1) = \left(1 + \mu_2(1 - e^{-s})\right)^{-N} \times M_{n,1}\left(\frac{1 + (\mu_2 - G_2)(1 - e^{-s})}{1 + \mu_2(1 - e^{-s})}, t_1\right) \tag{24}$$

with $N = c_i/a_i$, $n_{sp,i} = a_i/(a_i - b_i)$, $G_i = e^{(a_i - b_i)t_i}$ and $\mu_i = n_{sp,i}(G_i - 1)$, $i = 1, 2$.

This can be rewritten as (18) with $t = t_2 + t_1$, $\mu_0 = \mu_{0,1}G_2 = \tilde{\mu}_0 G_1 G_2$ and an expectation value $\mu = \mu_2 + \mu_1 G_2$ for the output noise in one mode. In the cascade

$$F_z = \frac{\mu_1 G_2 + \mu_2}{G_1 G_2} = F_{z1} + \frac{F_{z2}}{G_1} \tag{25}$$

holds. Recursion allows to obtain Friis' well known cascading formula

$$F = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2} + \cdots \tag{26}$$

for a larger cascade. This is important because optical trunk lines consist alternatively of amplifiers and attenuating fibers. A pure attenuator with $a = 0$, $b > 0$, $n_{sp} = 0$ has a noise figure $F = 1$ and a "gain" $G = e^{-bt} < 1$. For example, the cascade of an attenuator (index 1) and a subsequent amplifier (index 2) has the noise figure $F = 1 + \frac{F_2 - 1}{G_1}$. If the amplifier (index 1) is in front of the attenuator (index 2), $\mu_2 = 0$ holds. For increasing attenuation, $G_2 \to 0$, we find

$$M_{n,2}(e^{-s}, t_2 + t_1) = \left(1 + \mu_1 G_2(1 - e^{-s})\right)^{-N} e^{\frac{-\mu_{0,1}G_2(1-e^{-s})}{1+\mu_1 G_2(1-e^{-s})}} \approx e^{-\mu_{0,1}G_2(1-e^{-s})}\ . \tag{27}$$

This means a Laguerre becomes a Poisson distribution if attenuation is large. Since the Poisson distribution is also conserved under attenuation it has a special importance. An amplifier in which a, b and c vary as a function of the propagation coordinate may be subdivided into infinitesimal sections and treated by (26) in which summation is replaced by an integral

$$F = 1 + \int_0^t \left(\frac{d(F(\tau) - 1)}{dt}\right) G^{-1}(\tau)\,d\tau \tag{28}$$

with

$$\frac{d(F-1)}{dt} = n_{sp}(t)\frac{1 - G(t - dt)/G(t)}{dt} = \frac{a}{a-b}\,\frac{1 - e^{-(a-b)dt}}{dt} = a\ . \tag{29}$$

For constant coefficients, the integral

$$F = 1 + \int_0^t a(\tau)G^{-1}(\tau)\,d\tau = 1 + \int_0^t a(\tau)e^{-\int_0^\tau (a(\vartheta) - b(\vartheta))d\vartheta}\,d\tau \tag{30}$$

delivers just the known value $F = 1 + (1 - G^{-1})n_{sp}$. The longitudinal coordinate L inside the amplifier is found as $L = \int_0^t v_g\,d\tau$. If group velocity v_g is constant, $L = v_g t$, $dL = v_g\,dt$, the integration over time can be replaced by an integration over the longitudinal coordinate.

As an example, consider Raman amplification with a pump wavelength roughly 50–100 nm shorter than the signal wavelength. It can be used to render fibers lossless,

$a = b$. For a constant attenuation of 0.2 dB/km fiber "gain" without pumping ($a = 0$) would be $G = \mathrm{e}^{-bv_g^{-1}L} = 10^{-(1/10\,\mathrm{dB})0.2(\mathrm{dB/km})L}$. An $L = 50$ km long Raman fiber with $a = b$ ideally has the noise figure $F = 1 + bv_g^{-1}L = 3.3$ (if we ignore the difficulty of achieving $a = b$ over such large lengths). For comparison, an ideal amplifier ($n_{sp} = 1$) with $G = 10$ has $F = 1.9$. If it is placed at the input of a 50 km long fiber without Raman gain ($a = 0$) the cascade has a noise figure of 1.9. If it is placed at the fiber end the cascade noise figure is 10.

Noise figure measurement is easy The optical amplifier is operated without input signal. Gain saturation effects can be taken into account if the input signal is just switched off during short measurement intervals. Usually G is nearly polarization-insensitive, hence $p = 2$ polarization modes of a monomode fiber have to be taken into account. The output signal is passed through an optical filter with bandwidth B_o. Let the measured noise power be P. The quotient P/B_o equals the energy per mode in the frequency domain. The photon number is obtained after division by hf. The noise figure is therefore

$$F = 1 + \frac{P}{pB_o hfG} \quad . \tag{31}$$

If a polarizer is placed before the power meter only one polarization mode is evaluated and $p = 1$ holds.

Caution The hitherto prevailing noise figure definition [3] was based on photon number fluctuations (pnfs) in the limit of large photon numbers, $F_{\mathrm{pnf}} = G^{-1} + 2(1 - G^{-1})n_{sp}$. It is measured as $F_{\mathrm{pnf}} = \frac{1}{G}(1 + 2P/(pB_o hf))$. In an ideal amplifier it assumes the value 2, in an attenuator it has the value G^{-1}. It also fulfills the cascading formula (26)! However, it does not describe the noise of optical amplifiers exactly. By the way, it also differs from the classical microwave noise figure. Of course we can obtain our noise figure from the former one, $F = 1 + (F_{\mathrm{pnf}} - G^{-1})/2$.

4
Intensity distribution

Assume the photon distribution can be written as the so-called Poisson transform [2]

$$P(n) = \int_0^\infty p_x(x)\mathrm{e}^{-x}\frac{x^n}{n!}\,\mathrm{d}\tilde{x} \tag{32}$$

of a PDF $p_x(x)$ that belongs to a nonnegative RV x. If we vary the optical power, e.g., by changing the gain G of an optical amplifier, $P(n)$ and $p_x(x)$ will depend on G. Yet G mainly scales the x range so that the PDF $p_{\tilde{x}}(\tilde{x})$ of a normalized variable $\tilde{x} = x/G$ depends only weakly on G. From equal probabilities $p_x(x)\mathrm{d}x = p_{\tilde{x}}(\tilde{x})\mathrm{d}\tilde{x}$ it follows $p_x(x) = G^{-1}p_{\tilde{x}}(x/G)$. This allows to write the photon distribution as

$$P(n) = \int_0^\infty p_{\tilde{x}}(\tilde{x})\mathrm{e}^{-\tilde{x}G}\frac{(\tilde{x}G)^n}{n!}\,\mathrm{d}\tilde{x} \quad . \tag{33}$$

The MGF of $P(n)$ is $M_n(\mathrm{e}^{-s}) = \langle \mathrm{e}^{-sn}\rangle$. After inserting (33) the expression

$$\lim_{G\to\infty} M_n\left(\mathrm{e}^{-s/G}\right) = \lim_{G\to\infty}\sum_{n=0}^{\infty} P(n)\mathrm{e}^{-(s/G)n}$$

becomes

$$\lim_{G\to\infty} M_n\left(\mathrm{e}^{-s/G}\right) = M_{\tilde{x}}(\mathrm{e}^{-s}) \quad , \tag{34}$$

i.e., the MGF of the normalized variable $\tilde{x}$. Since MGFs can be calculated from PDFs and vice versa it is indeed possible to calculate $p_{\tilde{x}}(\tilde{x})$ from $P(n)$. For a Laguerre distribution MGF (18) we set $\tilde{\mu} = \lim_{G\to\infty}\mu G^{-1} = n_{sp}$, $\tilde{\mu}_0 = \mu_0 G^{-1}$, and find

$$M_{\tilde{x}}(\mathrm{e}^{-s}) = (1 + \tilde{\mu}s)^{-N}\,\mathrm{e}^{\frac{-\tilde{\mu}_0 s}{1+\tilde{\mu}s}} \quad . \tag{35}$$

The corresponding PDF is

$$p_{\tilde{x}}(\tilde{x}) = \frac{1}{\tilde{\mu}}\left(\frac{\tilde{x}}{\tilde{\mu}_0}\right)^{\frac{N-1}{2}} \mathrm{e}^{-\frac{\tilde{\mu}_0+\tilde{x}}{\tilde{\mu}}} I_{N-1}\left(\frac{2\sqrt{\tilde{x}\tilde{\mu}_0}}{\tilde{\mu}}\right) \quad , \tag{36}$$

a noncentral χ^2_{2N} (chi-square) PDF with $2N$ degrees-of-freedom ($\tilde{x} \geq 0$). For $\tilde{\mu}_0 = 0$, or if one starts from the central negative binomial distribution MGF (16), the MGF

$$M_{\tilde{x}}(\mathrm{e}^{-s}) = (1 + \tilde{\mu}s)^{-N} \tag{37}$$

of a central χ^2_{2N} or Gamma PDF

$$p_{\tilde{x}}(\tilde{x}) = \frac{1}{\Gamma(N)}\tilde{\mu}^N x^{N-1}\,\mathrm{e}^{-\tilde{x}/\tilde{\mu}} \quad . \tag{38}$$

is obtained. It is also found if one inserts $\lim_{|z|\to 0} I_\nu(z) = \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu$ into (36). Similar to the addition property of negative binomial distributions, the sum $x = x_1 + x_2$ of two statistically independent RVs x_1, x_2 with χ^2 PDFs and the same noise $\tilde{\mu} = n_{sp}$ is a new χ^2 PDF with $\tilde{\mu}_0 = \tilde{\mu}_{0,1} + \tilde{\mu}_{0,2}$, $N = N_1 + N_2$, as can be verified by multiplying the corresponding MGFs.

For comparison and in order to apply again (34) we assume a Poisson distribution with expectation value $\mu_0 = G\tilde{\mu}_0$ and find

$$M_{\tilde{x}}(\mathrm{e}^{-s}) = \mathrm{e}^{-\tilde{\mu}_0 s} \tag{39}$$

with corresponding PDF

$$p_{\tilde{x}}(\tilde{x}) = \delta(\tilde{x} - \tilde{\mu}_0) \quad . \tag{40}$$

This is a constant $\tilde{\mu}_0$. These findings justify assumption (32) and can be interpreted: when the light intensity is subject to a detection process, shot noise is added which is Poisson distributed for each value the intensity may assume with a certain probability density. The light intensity may carry just noise (central χ^2), signal and noise (noncentral χ^2), or just signal (constant distribution). The detection, modeled by the Poisson transform, results in central or noncentral negative binomial, or Poisson distributions, respectively. All these can be found in Table 1.

Central and noncentral negative binomial distributions with $2N = 4$ are plotted logarithmically for an ideal amplifier ($n_{sp} = 1$) in Fig. 1, but for easier comparison normalized probabilities $\log GP(nG)$ are displayed instead of $\log P(n)$. A value $\tilde{\mu}_0 = 81.4$ was chosen. The higher the gain $G = 1, 2, 4, \infty$, the higher the noise. In the noise-free case $G = 1$ Poisson distributions (o symbols) result. For infinite G χ_4^2 distributions result with a BER $= 10^{-9}$.

The intensity may be defined as the expectation value with respect to the detection process of the photon number. This expectation value normally is an RV itself with respect to ASE, which originates mainly from the amplifier input. Since normalization is arbitrary we may likewise understand the light intensity to be the expectation value of the squared electrical field magnitude, $\langle |\mathbf{E}|^2 \rangle$, including ASE noise.

A noncentral χ_{2N}^2 PDF (36) describes an RV $\chi_{2N}^2 \equiv x = \sum_{i=1}^{2N} x_i^2$, where x_i are Gaussian RVs with identical variances $\sigma_{x_i}^2 = n_{\mathrm{sp}}/2$. The square sum of the expectation values $\langle x_i \rangle = \sqrt{\mu_{0,i}}$ of z_i is $\mu_0 = \sum_{i=1}^{2N} \mu_{0,i}$. The electric field is therefore a carrier with constant amplitude, accompanied by Gaussian zero-mean noise x_i with $\sigma_{x_i}^2 = n_{\mathrm{sp}}/2$ in two quadratures,

$$\mathbf{E}(t) = \sqrt{2}\Big(\Big(\sqrt{\mu_0/M} + x_1\Big)\cos(\omega t) + x_2 \sin(\omega t)\Big)\mathbf{e}_1 \quad . \tag{41}$$

Here $\mathbf{e}_1$ is the unit vector of the signal polarization. For the time being, let $M = 1$. The intensity equals the expectation value of the photon number. One intensity sample is

$$\langle |\mathbf{E}(t)|^2 \rangle = \Big(\sqrt{\mu_0/M} + x_1\Big)^2 + x_2^2 \quad . \tag{42}$$

and contains 2 degrees-of-freedom. If the amplifier is polarization-independent, the orthogonal polarization $\mathbf{e}_2$ with $\mathbf{e}_1^+ \mathbf{e}_2 = 0$ also has to be considered,

$$\begin{aligned}\mathbf{E}(t) = \sqrt{2}\big(\big(\sqrt{\mu_0}/M + x_1\big)\cos(\omega t) + x_2 \sin(\omega t)\big)\mathbf{e}_1 \\ + (x_3 \cos(\omega t) + x_4 \sin(\omega t))\mathbf{e}_2 \quad ,\end{aligned} \tag{43}$$

$$\langle |\mathbf{E}(t)|^2 \rangle = \Big(\sqrt{\mu_0/M} + x_1\Big)^2 + x_2^2 + x_3^2 + x_4^2 \quad , \tag{44}$$

and there are 4 degrees-of-freedom.

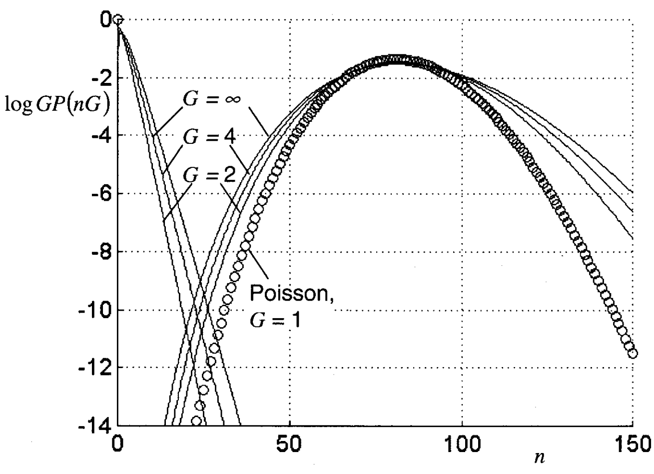

Fig. 1. Transition of normalized central and noncentral distributions from Poisson via negative binomial to χ_4^2

5 Application

We consider now an optical receiver (Fig. 2). Behind the optical receiver there is an optical filter with an impulse response which is a cosine oscillation having the frequency of the received signal and a rectangular envelope of duration $\tau_1 = 1/B_o$. B_o is the optical bandwidth of that filter. For constant signal statistically independent fields $\mathbf{E}(t + i\tau_1)$ are therefore obtained every time interval τ_1. After photodetection, statistically independent intensity samples $\langle |\mathbf{E}(t + i\tau_1)|^2 \rangle$ result. In our model the electrical part of the optical receiver should possess so little thermal noise that it is possible to give it an impulse response equal to a comb of $M = \tau_2/\tau_1$ Dirac pulses with equal amplitudes, spaced by τ_1. In that case the baseband filter is not a lowpass filter but has infinite bandwidth! The Dirac comb filter can in good approximation be exchanged against a filter with a continuous impulse response of length τ_2 or better $\sqrt{\tau_2^2 - \tau_1^2}$, unless M is very small. In order to avoid intersymbol interference τ_2 is of course chosen smaller than one bit duration T, but not much smaller since signal energy would otherwise be lost. Suitably normalized, the signal at the decision circuit input is therefore

$$\tilde{x} = \sum_{i=1}^{M} \langle |\mathbf{E}(t + i\tau_1)|^2 \rangle \tag{45}$$

with PDF (36). In the hypothetical, noise-free case $\langle \tilde{x} \rangle = \tilde{\mu}_0$ holds. The decision variable has $2N = 2pM$ degrees-of-freedom, where the number of polarizations is usually $p = 2$. For $\tilde{\mu}_0 = 0$ the PDF is given by (38).

Thermal noise in the electrical part of the optical receiver must also be taken into account. Figure 3 shows $\log P(n)$ for $G = 16, 64$ and 256, $\tilde{\mu}_0 = 100$ and $2N = 8$. However, Gaussian thermal noise with $\sigma = 523$ has been added which results in BER $= 7 \cdot 10^{-2}$, $1 \cdot 10^{-5}$, $2.5 \cdot 10^{-10}$, respectively. It is taken into account by convolving the probability distributions are convolved or multiplying the corresponding MGFs. A high gain is needed to make the relative thermal noise contribution insignificant. The chosen thermal noise $\sigma = 523$ corresponds to a receiver with a bandwidth of 7 GHz and a thermal noise of $10\,\mathrm{pA}/\sqrt{\mathrm{Hz}}$, suitable to receive 106 bit/s.

BER vs. $10\log(\tilde{\mu}_0/2)$, i.e., the mean photon number expressed in dB, has been calculated for $\tilde{\mu} = n_{\mathrm{sp}} = 1$, $G \to \infty$, i.e., χ_{2N}^2 PDFs. Fairly moderate penalties occur for rising N (Fig. 4). The BER curves become steeper as N increases.

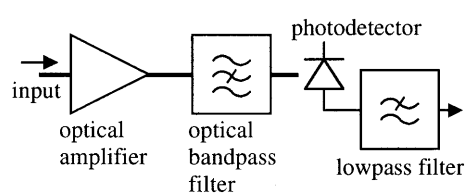

Fig. 2. Optical receiver. A polarizer could, but normally is not, placed behind the amplifier in order to block noise orthogonal to the signal

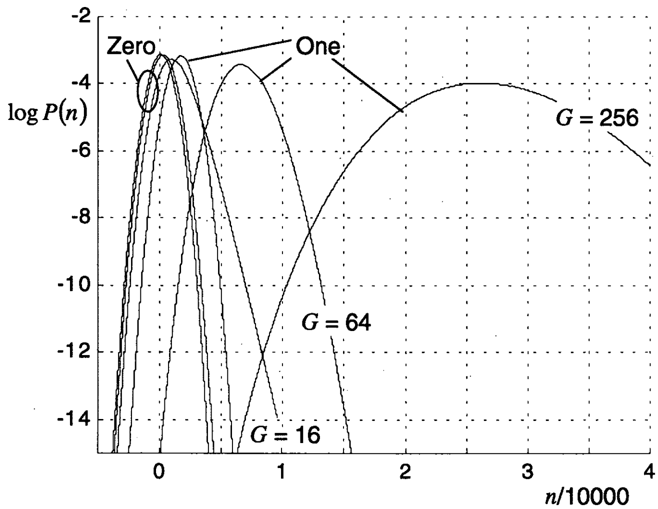

Fig. 3. Photon distributions with thermal noise added

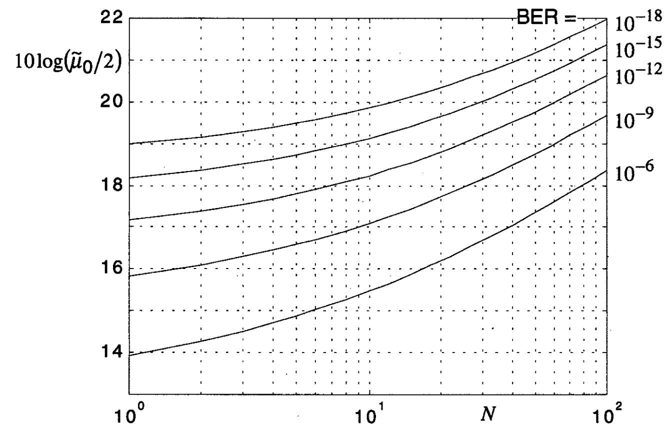

Fig. 5. BER vs. mean photon number with degrees-of-freedom as a parameter

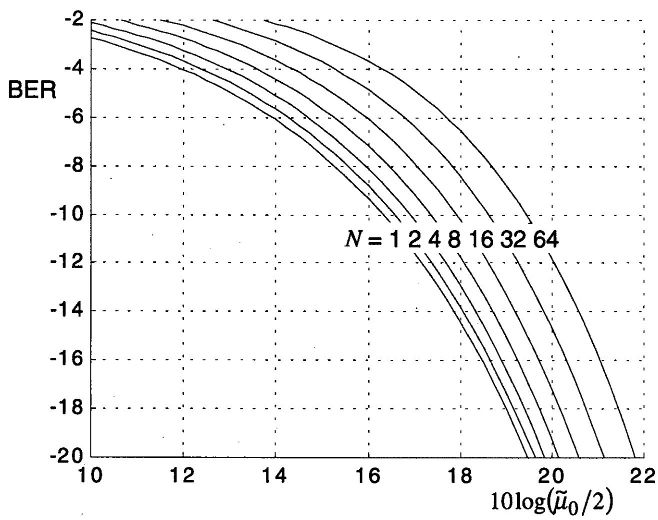

Fig. 4. BER vs. mean photon number with degrees-of-freedom as a parameter

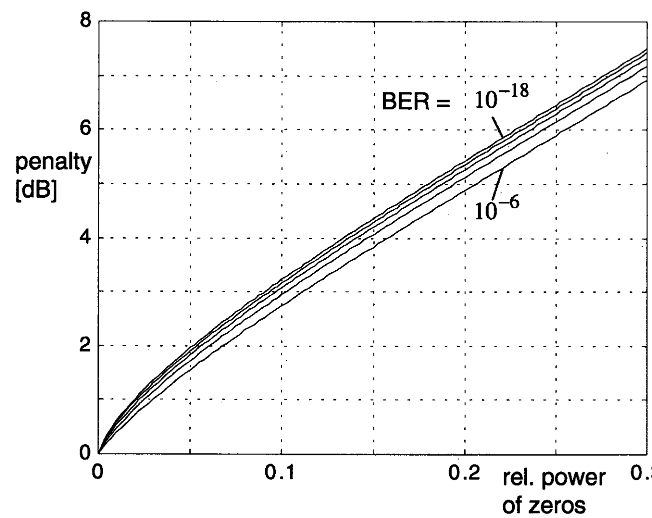

Fig. 6. Penalty of average signal power as a function of power ratio between zeros and ones

With more detail this behavior is investigated in Fig. 5. It shows $10\log(\tilde{\mu}_0/2)$ vs. N for several constant BER. If there is a finite power ratio between zeros and ones of the optical signal, a transmitted zero will also have a noncentral binomial or χ^2_{2N} distribution. For the latter case resulting penalties are displayed in Fig. 6.

6
Conclusion

Optical amplifiers are important building blocks in nowaday's optical communication system. The photons at the amplifier output obey noncentral or central negative binomial distributions for transmitted one or zero, respectively. If there is no gain, these become Poisson distributions. If the gain is infinite, these become chi-square distributions. The optical noise figure, correctly defined, is a direct measure for the photon distribution. Fairly moderate penalties occur if the optical filter bandwidth is larger than twice the electrical receiver bandwidth. Large penalties result if there is a bad extinction ratio between zeros and ones.

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