kT/C noise: Covariance matrix calculation of instationary noise in time-varying systems

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Abstract **The voltage variance at a capacitance C with a noisy resistance in parallel is kT/C, even if the resistance is infinite. This so-called kT/C noise may dominate in switched-capacitor circuits. In this paper kT/C noise is treated analytically, using explicit and implicit notations of differential equations. The resulting algorithms can be implemented in any circuit simulator with transient analysis, including cases where there are capacitor islands without capacitive paths to ground. The general approach allows to take into account high integrator bandwidth, slow switching and other methods to mediate kT/C noise. A numerical example shows that an extremely high integrator bandwidth is needed to reduce kT/C noise.**

1. Introduction

Fig. 1a shows a conductance G in parallel to a capacitance C. The thermal noise of conductance G is modeled by a noise current i with a two-sided power spectral density $S_{ii} = 2kTG$. The variance

$$s_{uu}(t) = \frac{1}{2\pi}\int \frac{2kTG}{G^2 + \omega^2 C^2} d\omega = \frac{kT}{C} \tag{1}$$

of voltage u is given by Parseval's theorem applied to the voltage spectral density which in turn is the current spectral density divided by the squared magnitude of the total admittance $G + j\omega C$. The result is $s_{uu}(t) = kT/C$, and $s_{qq}(t) = kTC$ is the variance of the charge q. Since no restrictions were made this result holds, simply speaking, also if the conductance

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vanishes, $G = 0$. Of course a $G > 0$ must have been present at some time in the past, which process will be investigated in this paper.

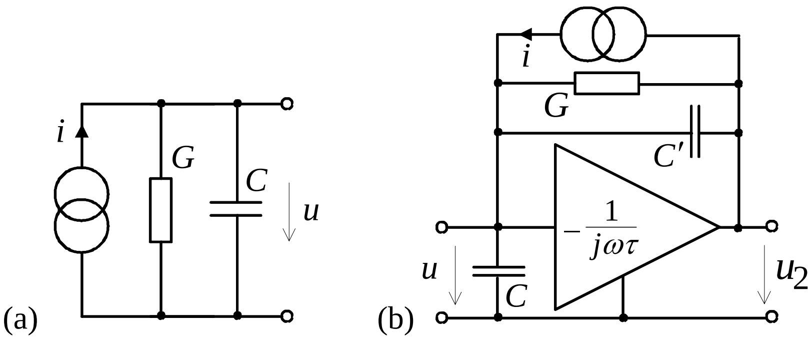

Fig. 1: Noisy conductance *G* across a capacitance (a), and charge amplifier (b)

In the current integrator or charge amplifier circuit of Fig. 1b with an inverting integrator the charge variance on the input node can be as low as

$$s_{qq}(t) = kTC' . \tag{2}$$

If $C' << C$, the charge variance can be substantially reduced in principle. However, the widely used equation (2) holds only for an infinite gain-bandwidth product of the integrator, $\tau \to 0$ in Fig. 1b. As will be seen later, the practically achievable noise reduction is quite limited.

In general the circuit components will be time-varying, and since the power spectral density of the noisy current source depends on the time-varying conductance *G* the noise is instationary.

A general expression for the calculation of the correlation function matrix in the time domain at the output of such a network has been given in [1]. A universal analytical treatment of instationary noise in time-varying networks has been given in [2] but calculation was restricted to cases where every circuit node with connection to a capacitor has a capacitive path to ground or to an independent voltage source node. If *C* is removed from Fig. 1b this condition is not met, which motivates the present work.

kT/C noise has been modeled thoroughly in [3] but effects such as finite switching time of switches were not taken into account, nor was a general solution given for implementation in a network simulator. Recent publications [4-6] show that kT/C noise continues to be a problem.

In this paper the covariance matrix in a time-varying network with white instationary noise will be expressed generally, and algorithms for its calculation along with the transient analysis of a network simulator will be developed, in particular for the (more complicated) implicit notation of the differential equations describing the system. Finally, an illustrative example will be given.

2. Analysis

The transfer of a signal vector $\mathbf{v}(t)$ through a linear, time-varying network with an impulse response matrix $\mathbf{h}(t,t_1)$ is described by

$$\mathbf{w}(t)=\int_{-\infty}^{\infty}\mathbf{h}(t,t_1)\mathbf{v}(t_1)dt_1 \tag{3}$$

where $\mathbf{w}(t)$ is a vector of output quantities [7]. The second argument (t_1) in $\mathbf{h}(t,t_1)$ is the time at which the system is excited, the first (t) is the time when the response is observed. Their difference $\tau=t-t_1$ is the age variable used in the impulse response matrix $\mathbf{h}(t-t_1)$ of a linear time-invariable system. The Fourier transform of the impulse response matrix $\mathbf{h}(t,t_1)$ with respect to the age variable is a time-variable transfer function matrix

$$\mathbf{H}(\omega,t)=\int_{-\infty}^{\infty}\mathbf{h}(t,t-\tau)e^{-j\omega\tau}d\tau\ . \tag{4}$$

The inversion formula is

$$\mathbf{h}(t,t-\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{H}(\omega,t)e^{j\omega\tau}d\omega\ . \tag{5}$$

Following the definitions [8, 9], the instationary correlation function matrix of signal vector $\mathbf{v}(t)$ is

$$\mathbf{s}_{vv}(\tau,t)=\left\langle\mathbf{v}(t)\mathbf{v}^{+}(t-\tau)\right\rangle . \tag{6}$$

Note that this differs from the definition in [1] (and (1), (2)) in so far as the first argument is a time difference. The brackets $\langle . \rangle$ symbolize the ensemble average, and this is why the result may still depend upon time *t*, not only on the time delay τ. The + is the hermitian conjugate operator. If the definition is applied to the output signal vector $\mathbf{w}(t)$ a straightforward calculation yields

$$\mathbf{s}_{ww}(\tau,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{h}(t,t_1)\mathbf{s}_{vv}(t_2,t_1)\mathbf{h}^{+}(t-\tau,t_1-t_2)dt_2 dt_1 . \tag{7}$$

With a slightly different notation this result has already been given in [1]. The relations between instationary correlation function ($\mathbf{s}(\tau,t)$) and spectrum ($\mathbf{S}(\omega,t)$) matrices are

$$\mathbf{S}(\omega,t) = \int_{-\infty}^{\infty} \mathbf{s}(\tau,t)e^{-j\omega\tau}d\tau , \quad \mathbf{s}(\tau,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{S}(\omega,t)e^{j\omega\tau}d\omega . \tag{8}$$

The scalar simplification of the correlation function matrix is the power spectral density. An important simplification results if the noise source is white. In this case,

$$\mathbf{s}_{vv}(\tau,t) = \delta(\tau)\mathbf{S}_{vv}(t) \tag{9}$$

where $\delta(\tau)$ is a Dirac impulse and $\mathbf{S}_{vv}(t)$ is the white, instationary correlation spectrum (matrix) of the input signal $\mathbf{v}(t)$. Note that even though the Fourier transform of a stochastic $\mathbf{v}(t)$ does not exist the correlation function matrix and hence correlation spectrum do exist.

The correlation spectrum matrix can be determined using Russer’s fundamental noise analysis techniques [9]. In particular, $\mathbf{S}_{ww}(\omega) = \mathbf{H}(\omega)\mathbf{S}_{vv}(\omega)\mathbf{H}^{+}(\omega)$ holds if $\mathbf{W}(\omega) = \mathbf{H}(\omega)\mathbf{V}(\omega)$ is true for deterministic, or temporally truncated stochastic input signals. If $\mathbf{Y}(\omega)$ is a passive, time-invariable admittance matrix the stationary current correlation spectrum is

$$\mathbf{S}_{ii}(\omega) = kT\left(\mathbf{Y}(\omega) + \mathbf{Y}^{+}(\omega)\right). \tag{10}$$

With (9) the correlation function matrix of the output signal simplifies to

$$\mathbf{s}_{ww}(\tau,t) = \int_{-\infty}^{\infty} \mathbf{h}(t,t_1)\mathbf{S}_{vv}(t_1)\mathbf{h}^{+}(t-\tau,t_1)dt_1 . \tag{11}$$

If the delay is zero, $\tau = 0$, we obtain the instationary covariance matrix

$$\mathbf{s}_{ww}(t) \equiv \mathbf{s}_{ww}(0,t) = \int_{-\infty}^{\infty} \mathbf{h}(t,t_1)\mathbf{S}_{vv}(t_1)\mathbf{h}^+(t,t_1)dt_1 \,. \tag{12}$$

Note that the first argument 0 of the covariance matrix is dropped here and in the following.

In many cases the switching action can be modeled to be instantaneous. This means that during certain time periods the impulse responses are time-invariable and the noise is stationary. In these cases (12) simplifies to

$$\mathbf{s}_{ww} = \int_{-\infty}^{\infty} \mathbf{h}(t-t_1)\mathbf{S}_{vv}\,\mathbf{h}^+(t-t_1)dt_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{H}(\omega)\mathbf{S}_{vv}\,\mathbf{H}^+(\omega)d\omega \,. \tag{13}$$

If the switch stays in its position only between t' and t we can write

$$\mathbf{s}_{ww}(t) = \int_{t'}^{t} \mathbf{h}(t-t_1)\mathbf{S}_{vv}\,\mathbf{h}^+(t-t_1)dt_1 + \mathbf{s}_{ww}(t'+0). \tag{14}$$

This allows integration piecewise, provided switching at t' does not change the covariance matrix, $\mathbf{s}_{ww}(t'+0) = \mathbf{s}_{ww}(t'-0)$.

We return now to the general case. Although much simpler than (7), equation (12) is still cumbersome to evaluate directly since the value of the time-variable impulse response matrix has to be determined at one point (t) for all t_1. Note that the impulse response is normally not determined during a transient response analysis of a circuit.

However, the calculation can be carried out much more efficiently. Assume the network can be modeled by a system

$$\begin{aligned} \mathbf{x}_{n+1} &= \tilde{\mathbf{A}}_n \mathbf{x}_n + \tilde{\mathbf{B}}_n \mathbf{v}_n \\ \mathbf{w}_n &= \tilde{\mathbf{C}}_n \mathbf{x}_n + \tilde{\mathbf{D}}_n \mathbf{v}_n \end{aligned} \tag{15}$$

of time-discrete state equations, where $\mathbf{v}_n$ is an input, $\mathbf{x}_n$ a state, and $\mathbf{w}_n$ an output vector. The state vector contains all independent charges and magnetic fluxes. State vector $\mathbf{x}_n$ depends only on past input vectors $\mathbf{v}_m$ $(m < n)$. Since we assume white noise, $\mathbf{v}_n$ and $\mathbf{v}_m$ $(m < n)$ are uncorrelated, and so are $\mathbf{v}_n$ and $\mathbf{x}_n$. If the covariance matrices of (15) are calculated the mixed terms on the right sides therefore disappear, and the simple result

$$\begin{aligned} \mathbf{s}_{xx,n+1} &= \tilde{\mathbf{A}}_n \mathbf{s}_{xx,n} \tilde{\mathbf{A}}_n^+ + \tilde{\mathbf{B}}_n \mathbf{s}_{vv,n} \tilde{\mathbf{B}}_n^+ \\ \mathbf{s}_{ww,n} &= \tilde{\mathbf{C}}_n \mathbf{s}_{xx,n} \tilde{\mathbf{C}}_n^+ + \tilde{\mathbf{D}}_n \mathbf{s}_{vv,n} \tilde{\mathbf{D}}_n^+ \end{aligned} \tag{16}$$

is obtained. The absolute time (t) argument has been dropped and is replaced by an index in the subscript. For discrete time steps δt, the white noise covariance matrix is

$$\mathbf{s}_{vv,n} = \delta t^{-1} \mathbf{S}_{vv,n} . \tag{17}$$

In the case of noise currents the instationary correlation matrix $\mathbf{S}_{vv,n}$ is given by (10), with an index rather than a time variable. Successive indexes n represent successive time variables spaced by δt each.

As the next case, consider the calculation of charge covariance matrix $\mathbf{s}_{qq,n}$ and voltage covariance matrix $\mathbf{s}_{uu,n}$ in a circuit simulator. The voltage-dependent capacitances are taken into account by

$$\mathbf{q}_n = \hat{\mathbf{Q}}_n(\mathbf{u}_n) \tag{18}$$

where $\hat{\mathbf{Q}}_n$ is a nonlinear, time-variable function. Vector $\mathbf{u}_n$ represents all node voltages. Let us assume that the charge vector $\mathbf{q}_n$ represents only the number of independent connections to capacitive elements, which may be smaller than the number of nodes with connected capacitive elements. The set of charges needed for further description of the network, usually all node charges, is obtained by multiplication of an incidence matrix $\mathbf{K}_n$ by vector $\mathbf{q}_n$. E.g., matrix $\mathbf{K}_n$ may contain a 1 wherever an independent charge (≅ column) is a node charge (≅ line). One particular node (≅ line) of any capacitance island having no connection to ground carries a -1 for all other, independent charges (≅ columns) present at this island. The remaining elements of $\mathbf{K}_n$ are zero. $\mathbf{K}_n$ may be time-variable but will normally be fixed. As a simple example, a single capacitor with no connection to ground is connected to two nodes from which it draws identical but oppositely poled charging currents, but is characterized by just one charge.

The node currents are expressed by

$$\mathbf{i}_n = \hat{\mathbf{I}}_n(\mathbf{u}_n) + \mathbf{K}_n \dot{\mathbf{q}}_n . \tag{19}$$

Here $\mathbf{i}_n$ is the node current vector, $\dot{\mathbf{q}}_n$ the vector of independent currents flowing into the capacitors, and $\hat{\mathbf{I}}_n$ is another nonlinear, time-variable vector function which represents the currents flowing into the conductive circuit elements. In contrast to (15) the equations (18), (19) are implicit because (18) usually can not be inverted, not even in the linear case.
After linearization around the operation point, $\hat{\mathbf{Q}}_n$, $\hat{\mathbf{I}}_n$ reduce to a matrix of capacitances $\mathbf{C}_n$ and a matrix of conductances $\mathbf{G}_n$:

$$\mathbf{i}_n = \mathbf{G}_n \mathbf{u}_n + \mathbf{K}_n \dot{\mathbf{q}}_n \tag{20}$$

$$\mathbf{q}_n = \mathbf{C}_n \mathbf{u}_n \tag{21}$$

The dependent variables are now small-signal quantities such as noise. It is possible to solve (20), (21) as a function of $\mathbf{i}_n$: E.g., using $\dot{\mathbf{q}}_n = \delta t^{-1}(\mathbf{q}_n - \mathbf{q}_{n-1})$, the solution $\mathbf{u}_n = \left(\mathbf{G}_n + \mathbf{K}_n \delta t^{-1} \mathbf{C}_n\right)^{-1}\left(\mathbf{i}_n + \mathbf{K}_n \delta t^{-1} \mathbf{C}_{n-1} \mathbf{u}_{n-1}\right)$ is obtained, from which $\mathbf{q}_n$ and $\dot{\mathbf{q}}_n$ are derived. Voltage vector $\mathbf{u}_{n-1}$ is known from the previous integration step. The inhomogeneous solution

$$\mathbf{u}_n = \mathbf{u}_{n,q} + \mathbf{u}_{n,i}\,, \tag{22}$$

$$\mathbf{q}_n = \mathbf{q}_{n,q} + \mathbf{q}_{n,i}\,, \tag{23}$$

is a superposition of a homogeneous solution (index *q*) which is due to the charges alone, without currents,

$$\begin{aligned} \mathbf{0} &= \mathbf{G}_n \mathbf{u}_{n,q} + \mathbf{K}_n \dot{\mathbf{q}}_{n,q} \\ \mathbf{q}_n &= \mathbf{C}_n \mathbf{u}_{n,q} \end{aligned}\,, \tag{24}$$

and a particular solution (index *i*) which is due to the currents alone, without charges,

$$\begin{aligned} \mathbf{i}_n &= \mathbf{G}_n \mathbf{u}_{n,i} + \mathbf{K}_n \dot{\mathbf{q}}_{n,i} \\ \mathbf{0} &= \mathbf{C}_n \mathbf{u}_{n,i} \end{aligned}\,. \tag{25}$$

If we approximate

$$\dot{\mathbf{q}}(t) = \frac{\mathbf{q}_{n+1} - \mathbf{q}_n}{\delta t} \tag{26}$$

and solve for $\mathbf{q}_{n+1}$, (23) becomes

$$\mathbf{q}_{n+1} = \left(\mathbf{q}_n + \delta t \dot{\mathbf{q}}_{n,q}\right) + \delta t \dot{\mathbf{q}}_{n,i} \tag{27}$$

Here the solutions of (24), (25) need to be inserted. The first summand on the right side depends only on past noise samples $\mathbf{i}_h$ ($h < n$) and the second summand only on the present noise $\mathbf{i}_n$. For white noise the summands are therefore uncorrelated and the covariance function is

$$\begin{aligned} &\mathbf{s}_{qq,n+1} = T_q + T_i \\ &T_q = \left\langle \left(\mathbf{q}_n + \delta t \dot{\mathbf{q}}_{n,q}\right)\left(\mathbf{q}_n + \delta t \dot{\mathbf{q}}_{n,q}\right)^+ \right\rangle \qquad T_i = \delta t^2 \left\langle \dot{\mathbf{q}}_{n,i} \dot{\mathbf{q}}_{n,i}^+ \right\rangle . \end{aligned} \tag{28}$$

Term T_q can be calculated from $\mathbf{s}_{qq,n} = \left\langle \mathbf{q}_n \mathbf{q}_n^+ \right\rangle$ as follows: Charge vector $\mathbf{q}_n$ of length k can be written as the sum of k fixed charge vectors $\mathbf{q}_{n,j}$ ($j = 1...k$) times uncorrelated Gaussian, zero-mean, unit-variance random variables $r_{n,q,j}$. When arranged as the columns of a matrix $\left[\mathbf{q}_{n,j}\right] = \left[\mathbf{q}_{n,1}, \mathbf{q}_{n,2}, ..., \mathbf{q}_{n,k}\right]$ and a vector $\mathbf{r}_{n,q} = \left[r_{n,q,1}, r_{n,q,2}, ..., r_{n,q,k}\right]^+$, respectively, these quantities combine as

$$\mathbf{q}_n = \left[\mathbf{q}_{n,j}\right] \mathbf{r}_{n,q} \quad . \tag{29}$$

Taking $\left\langle \mathbf{r}_{n,q} \mathbf{r}_{n,q}^+ \right\rangle = \mathbf{1}$ into account the correlation matrix is found to be

$$\mathbf{s}_{qq,n} = \left[\mathbf{q}_{n,j}\right]\left[\mathbf{q}_{n,j}\right]^+ . \tag{30}$$

On the other hand the covariance matrix $\mathbf{s}_{qq,n}$ is real, symmetric and positive semidefinite which allows diagonalization with nonnegative eigenvalues $\lambda_{q,j}$ and an orthogonal eigenvector matrix $\mathbf{A}_q$,

$$\mathbf{s}_{qq,n} = \mathbf{A}_q \operatorname{diag}\left(\lambda_{q,j}\right) \mathbf{A}_q^+ . \tag{31}$$

Comparison of (30), (31) yields

$$\left[\mathbf{q}_{n,j}\right] = \mathbf{A}_q \operatorname{diag}\left(\sqrt{\lambda_{q,j}}\right). \tag{32}$$

Due to the linearization each charge vector $\mathbf{q}_{n,j}$ corresponds to a specific charging current $\dot{\mathbf{q}}_{n,q,j}$. These currents can be found by solving (24) as described above, and can be arranged in a matrix $\left[\dot{\mathbf{q}}_{n,q,j}\right] = \left[\dot{\mathbf{q}}_{n,q,1}, \dot{\mathbf{q}}_{n,q,2}, ..., \dot{\mathbf{q}}_{n,q,k}\right]$. Insertion of (29) and

$$\dot{\mathbf{q}}_{n,q} = \left[\dot{\mathbf{q}}_{n,q,j}\right] \mathbf{r}_{n,q} \tag{33}$$

into (28) allows to evaluate

$$T_q = \left(\left[\mathbf{q}_{n,j}\right] + \delta t\left[\dot{\mathbf{q}}_{n,q,j}\right]\right)\left(\left[\mathbf{q}_{n,j}\right] + \delta t\left[\dot{\mathbf{q}}_{n,q,j}\right]\right)^+ . \tag{34}$$

A similar calculation yields

$$T_i = \delta t^2 \left[\dot{\mathbf{q}}_{n,i,l}\right]\left[\dot{\mathbf{q}}_{n,i,l}\right]^+ . \tag{35}$$

Here $l = 1...m$ where m is the length of current noise vector $\mathbf{i}_n$ and random vector $\mathbf{r}_{n,i}$. With $\mathbf{i}_n = \left[\mathbf{i}_{n,l}\right]\mathbf{r}_{n,i}$, $\left[\mathbf{i}_{n,l}\right] = \mathbf{A}_i \operatorname{diag}\left(\sqrt{\lambda_{i,l}}\right)$, $\mathbf{s}_{ii,n} = \left[\mathbf{i}_{n,l}\right]\left[\mathbf{i}_{n,l}\right]^+ = \mathbf{A}_i \operatorname{diag}\left(\lambda_{i,l}\right)\mathbf{A}_i^+$ the columns $\dot{\mathbf{q}}_{n,i,l}$ of matrix $\left[\dot{\mathbf{q}}_{n,i,l}\right] = \left[\dot{\mathbf{q}}_{n,i,1}, \dot{\mathbf{q}}_{n,i,2}, ..., \dot{\mathbf{q}}_{n,i,m}\right]$ are obtained from $\mathbf{i}_{n,l}$ by solving (25). Vectors $\mathbf{q}_n$ and $\mathbf{i}_n$ are uncorrelated, $\left\langle \mathbf{r}_{n,q}\mathbf{r}_{n,i}{}^+ \right\rangle = \mathbf{0}$. Note that $\mathbf{s}_{ii,n} = \delta t^{-1}\mathbf{S}_{ii,n}$.

Voltage covariance matrices

$$\begin{aligned} \mathbf{s}_{uu,q,n} &= \left[\mathbf{u}_{n,q,j}\right]\left[\mathbf{u}_{n,q,j}\right]^+ \\ \mathbf{s}_{uu,i,n} &= \left[\mathbf{u}_{n,i,l}\right]\left[\mathbf{u}_{n,i,l}\right]^+ \\ \mathbf{s}_{uu,n} &= \mathbf{s}_{uu,q,n} + \mathbf{s}_{uu,i,n} \end{aligned} \tag{36}$$

can also be calculated where $\left[\mathbf{u}_{n,q,j}\right] = \left[\mathbf{u}_{n,q,1}, \mathbf{u}_{n,q,2}, ..., \mathbf{u}_{n,q,k}\right]$, $\left[\mathbf{u}_{n,i,l}\right] = \left[\mathbf{u}_{n,i,1}, \mathbf{u}_{n,i,2}, ..., \mathbf{u}_{n,i,m}\right]$ are the voltage solutions of (24), (25). Only $\mathbf{s}_{qq,n}$ and $\mathbf{s}_{uu,q,n}$ represent kT/C noise, having smooth correlation functions in the time domain. Covariance matrix $\mathbf{s}_{uu,i,n}$ corresponds to white noise, and its instationary correlation function has, for continuous time, the form $\mathbf{s}_{uu,i}(\tau, t) = \delta(\tau)\,\mathbf{S}_{uu,i}(t)$.

If the node charges are all independent it is possible to choose $\mathbf{K}_n = \mathbf{1}$, the capacitance matrix $\mathbf{C}_n$ is quadratic and can be inverted, and $\mathbf{u}_{n,i} = \mathbf{0}$, $\mathbf{u}_n = \mathbf{u}_{n,q}$ hold. This case is covered by [2] (in a different notation). Equations (21), (27), (28), (36) can then be rewritten as

$$\begin{aligned} \mathbf{q}_{n+1} &= \left(\mathbf{1} - \delta t \mathbf{G}_n \mathbf{C}_n^{-1}\right)\mathbf{q}_n + \delta t \mathbf{i}_n \\ \mathbf{u}_n &= \mathbf{C}_n^{-1}\mathbf{q}_n \end{aligned} , \tag{37}$$

$$\begin{aligned} \mathbf{s}_{qq,n+1} &= \left(\mathbf{1} - \delta t \mathbf{G}_n \mathbf{C}_n^{-1}\right)\mathbf{s}_{qq,n}\left(\mathbf{1} - \delta t \mathbf{G}_n \mathbf{C}_n^{-1}\right)^+ + \delta t^2 \mathbf{s}_{ii,n} \\ \mathbf{s}_{uu,n} &= \mathbf{C}_n^{-1}\mathbf{s}_{qq,n}\mathbf{C}_n^{-1+} \end{aligned} . \tag{38}$$

3. Example

In the circuit of Fig. 1b instantaneous off-switching (i.e., removal of G) will not alter the voltages across the capacitors, but a relaxation process after off-switching will. Therefore it is convenient to choose the charge

$$q = \int \left(i + G(u_2 - u)\right) dt = \left(C + C'\right)u - C'u_2 \tag{39}$$

at the input node as the output quantity **w**. A standard analysis reveals

$$H(\omega) = \frac{Q}{I} = \frac{C' + j\omega\tau\left(C' + C\right)}{G + j\omega\left(G\tau + C'\right) + \left(j\omega\right)^2 \tau\left(C' + C\right)}\,. \tag{40}$$

If only the feedback conductance G is noisy then the correlation matrix with respect to the port where the current source is connected is a scalar, $S_{ii} = 2kTG$. Evaluation of (13) in the frequency domain results in

$$s_{qq}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|H(\omega)\right|^2 S_{ii}\, d\omega = kTC_{eff} \qquad C_{eff} = C' + \frac{G\tau C}{G\tau + C'}\,. \tag{41}$$

This is the charge variance at the input node if the switch has remained closed (i.e., G has been present) for a sufficiently long time. For $\frac{1}{G\tau} >> \frac{1}{C'}\left(\frac{C}{C'} - 1\right)$, (2) is obtained.

In the time domain the system is governed by

$$\begin{aligned} i &= G\left(u - u_2\right) + \left(C + C'\right)\dot{u} - C'\dot{u}_2 \\ 0 &= \tau^{-1}u + \dot{u}_2 \end{aligned} \tag{42}$$

or, equivalently,

$$\begin{aligned} \begin{bmatrix} i \\ 0 \end{bmatrix} &= \begin{bmatrix} G & -G \\ \tau^{-1} & 0 \end{bmatrix}\begin{bmatrix} u \\ u_2 \end{bmatrix} + \begin{bmatrix} \dot{q} \\ \dot{u}_2 \end{bmatrix} \\ \begin{bmatrix} q \\ u_2 \end{bmatrix} &= \begin{bmatrix} C + C' & -C' \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u \\ u_2 \end{bmatrix} \end{aligned}\,. \tag{43}$$

The latter expression corresponds to (20), (21) although the admittance matrix of Fig. 1b does not exist. Quantity q is again the charge at the input node. The correlation matrix of the noise quantities $\begin{bmatrix} i \\ 0 \end{bmatrix}$ is

$$\mathbf{S}_{ii,n} = \begin{bmatrix} 2kTG & 0 \\ 0 & 0 \end{bmatrix}. \tag{44}$$

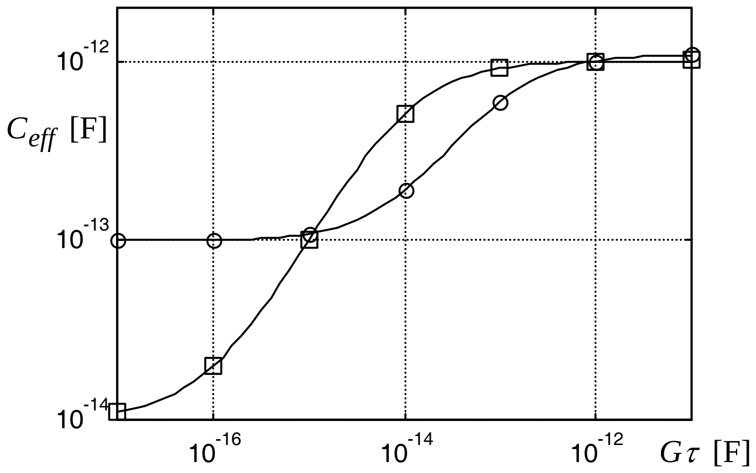

Fig. 2: Effective kT/C capacitance C_{eff} for the circuit of Fig. 1b as a function of $G\tau$. Circles $C' = 10^{-13}$ F; squares $C' = 10^{-14}$ F.

Software has been written to implement (28) for the example of (43), (44). The element (1,1) of $\mathbf{s}_{qq,n}$ is the charge variance; when divided by kT it becomes C_{eff} of (41). Fig. 2 compares the time-domain evaluation (symbols) with the results of (41) (solid lines). The technically accessible products $G\tau$ of noisy conductance G times integrator time constant τ are in the right part of the figure. Only small reductions of kT/C noise can presently be expected from the integrator circuit of Fig. 1b.

Of course the frequency- and time-domain results are identical. The advantage of the time-domain evaluation is that any influence not present in (41), e.g., finite charge integration time, slow switching and means to mediate kT/C noise can be taken into account exactly during the transient analysis of a circuit simulator.

Higher-order integration schemes rather than (27) have also been tried for the implementation of (28). As expected they have allowed to use larger integration step sizes.

4. Conclusions

A method for evaluation of kT/C noise in electronic circuits has been presented. Other than in [2], the general case of capacitor islands with no capacitive connection to ground is included because the differential equations describing the network can be solved in implicit notation. The method is suitable for implementation in circuit simulators. It allows to take into account high integrator bandwidth, slow switching and any other method to mediate kT/C noise.

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